d'Alembert solution of the wave equation.

Edwards and Penney have a typo in the d'Alembert solution (equations (37) and (39) on page 639 in section 9.6). This is an easier way to derive the solution.

Suppose we have the wave equation

$$u_{tt} = a^2 u_{xx}. \tag{1}$$

And we wish to solve the equation (1) given the conditions

$$u(0,t) = u(L,t) = 0 \quad \text{for all } t, \tag{2}$$
$$u(x,0) = f(x) \quad 0 < x < L, \tag{3}$$
$$u_t(x,0) = g(x) \quad 0 < x < L. \tag{4}$$

We will transform the equation into a simpler form where it can be solved by simple integration. We change variables to $\xi = x - at$, $\eta = x + at$ and we use the chain rule:

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial x} &= \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} = -a^2\frac{\partial}{\partial \xi} + a^2\frac{\partial}{\partial \eta} \end{aligned}$$

We compute

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right) = \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2} \\ \frac{\partial^2 u}{\partial t^2} &= \left(-a^2\frac{\partial}{\partial \xi} + a^2\frac{\partial}{\partial \eta}\right)\left(-a^2\frac{\partial u}{\partial \xi} + a^2\frac{\partial u}{\partial \eta}\right) = a^2\frac{\partial^2 u}{\partial \xi^2} - 2a^2\frac{\partial^2 u}{\partial \xi \partial \eta} + a^2\frac{\partial^2 u}{\partial \eta^2} \end{aligned}$$

Then

$$0 = a^2 u_{xx} - u_{tt} = 4a^2\frac{\partial^2 u}{\partial \xi \partial \eta}$$

And therefore the wave equation (1) transforms into $u_{\xi\eta} = 0$. It is easy to find the general solution to this equation by integration twice. First suppose you integrate with respect to η and notice that the constant of integration depends on ξ to get $u_\xi = C(\xi)$. Now integrate with respect to ξ and notice that the constant of integration must depend on η. Thus, $u = \int C(\xi)d\xi + B(\eta)$. The solution must then be of the following form for some functions A and B.

$$u = A(\xi) + B(\eta) = A(x - at) + B(x + at).$$

We will need to solve for the given conditions. First let $F(x)$ denote the *odd extension* of $f(x)$ and $G(x)$ denote the *odd extension* of $g(x)$. We let

$$A(x) = \frac{1}{2}F(x) - \frac{1}{2a}\int_0^x G(s)\ ds \qquad B(x) = \frac{1}{2}F(x) + \frac{1}{2a}\int_0^x G(s)\ ds.$$

The solution is explicitly

$$\begin{aligned} u(x,t) &= \frac{1}{2}F(x-at) - \frac{1}{2a}\int_0^{x-at} G(s)\ ds + \frac{1}{2}F(x+at) + \frac{1}{2a}\int_0^{x+at} G(s)\ ds \\ &= \frac{F(x-at) + F(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} G(s)\ ds \end{aligned}$$

Checking our work.

Let us check that this works. So

$$u(x,0) = \frac{1}{2}F(x) - \frac{1}{2a}\int_0^x G(s)\ ds + \frac{1}{2}F(x) + \frac{1}{2a}\int_0^x G(s)\ ds = F(x).$$

So far so good. Assume for simplicity F is differentiable. By the fundamental theorem of calculus we have

$$u_t(x,t) = \frac{-a}{2}F'(x-at) + \frac{1}{2}G(x-at) + \frac{a}{2}F'(x+at) + \frac{1}{2}G(x+at)$$

So

$$u_t(x,0) = \frac{-a}{2}F'(x) + \frac{1}{2}G(x) + \frac{a}{2}F'(x) + \frac{1}{2}G(x) = G(x).$$

Yay! We're smoking now. OK, now the boundary conditions. Note that F and G are odd. Also $\int_0^x G(s)ds$ is an even function of x because G is odd (do the substitution $s = -v$ to see that). So

$$\begin{aligned} u(0,t) &= \frac{1}{2}F(-at) - \frac{1}{2a}\int_0^{-at} G(s)\ ds + \frac{1}{2}F(at) + \frac{1}{2a}\int_0^{at} G(s)\ ds \\ &= \frac{-1}{2}F(at) - \frac{1}{2a}\int_0^{at} G(s)\ ds + \frac{1}{2}F(at) + \frac{1}{2a}\int_0^{at} G(s)\ ds = 0 \end{aligned}$$

Now F and G are $2L$ periodic as well. Furthermore

$$\begin{aligned} u(L,t) &= \frac{1}{2}F(L-at) - \frac{1}{2a}\int_0^{L-at} G(s)\ ds + \frac{1}{2}F(L+at) + \frac{1}{2a}\int_0^{L+at} G(s)\ ds \\ &= \frac{1}{2}F(-L-at) - \frac{1}{2a}\int_0^{L} G(s)\ ds - \frac{1}{2a}\int_0^{-at} G(s)\ ds + \frac{1}{2}F(L+at) + \frac{1}{2a}\int_0^{L} G(s)\ ds + \frac{1}{2a}\int_0^{at} G(s)\ ds \\ &= \frac{-1}{2}F(L+at) - \frac{1}{2a}\int_0^{at} G(s)\ ds + \frac{1}{2}F(L+at) + \frac{1}{2a}\int_0^{at} G(s)\ ds = 0 \end{aligned}$$

Notes

It is best to memorize the procedure rather than the formula itself. You should remember that a solution to the wave equation is a superposition of two waves traveling at opposite directions. That is

$$u(x,t) = A(x-at) + B(x+at).$$

If you think about it, the formulas for A and B are then not hard to guess. Also note that when $g(x) = 0$ (and hence $G(x) = 0$) we have

$$u(x,t) = \frac{F(x-at) + F(x+at)}{2}$$

Here is where the book got it wrong. If you let

$$H(x) = \int_0^x G(s)ds,$$

then assuming that $F(x) = 0$ the solution is

$$\frac{-H(x-at) + H(x+at)}{2a}.$$

So by superposition we get a solution in the general case when neither f nor g are identically zero.

$$u(x,t) = \frac{F(x-at) + F(x+at)}{2} + \frac{-H(x-at) + H(x+at)}{2a},$$

which is what the book was going for but it missed the minus sign.

Warning: Make sure you use the odd extensions F and G, when you have formulas for f and g. The thing is, those formulas in general hold only for $0 < x < L$, and are note equal to F and G for other x.